CAHIERS MAGELLANES-NS
Volume 06 Issue 2
2024

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310310
http://magellanes.com/

ZERO-STABILITY AND CONVERGENCE FOR INITIAL VALUE PROBLEMS

Hussein Jamil Hamayd Al-Ghaz[1] , Ali Sehen Athab[2]

1.National University of Science and Technology.Iraq
2. Thi-Qar general directorate of education. Iraq

Corresponding author : Hussein Jamil Hamayd Al-Ghaz

Abstract:
Within this research, we worked on the concepts of zero stability and convergence for initial value problems via numerical methods, and the primary goal was to analyze the convergence properties of one-step methods, with a direct and special focus on Euler's method, within the field of linear problems, as this paper that we conducted investigates error. Local truncation and the global error of Euler's method, in order to extract ideas about its accuracy and asymptotic behavior, where a general error term is derived based on the maximum local truncation error and the size of the time step, while mentioning some studies that develop a broader analysis of other one-step methods and nonlinear problems, and through these results we arrive at To a deeper understanding of numerical methods for initial value problems thereby paving the way for further research and progress in this field.

Keywords: zero invariance, convergence, initial value problems, numerical methods, Euler's method, linear problems, local truncation error, global error, one-step methods, nonlinear problems

Introduction:

We must realize that the numerical approximation of initial value problems is of very great importance in various scientific and engineering disciplines, and within the research we focus on the concepts of zero stability and convergence by solving initial value problems through numerical methods. Our goal is to analyze the convergence properties of one-step methods and model them, with a direct and special focus on Euler's method for linear problems [1-2]. We delved into the concept of local truncation error with and global error through convergence analysis of Euler's method, where one-step methods come into play. The biggest example is Euler's method, which takes a crucial role in approximating the initial value solution. The problem was that Euler's method is a first-order numerical method and approximates the solution is by setting an interval in addition to using the direct approximation of the forward difference of the derivative. This method is represented by the following equation:

$$\boldsymbol{y_{\{n+1\}} = y_n + h * f\,(t_n, y_n)(1) Here, y_{\{n+1\}}}$$

It represents the numerical approximation of the solution over the next time step, and y_n represents the numerical approximation at the actual current time step. In addition, h will indicate the step size, and f (t_n, y_n) is a numerical representation of the derivative over time t_n through y_n. By analyzing the convergence properties of Euler's method, we can evaluate its accuracy and reliability in

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310310
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approximating the true solution to initial value problems.
Convergence analysis will examine the behavior of the method when the step size approaches zero very closely. We will also measure the local truncation error, the extent to which the method deviates from the exact solution at each time step, but while the global error determines the cumulative deviation across multiple time steps,... Through it, it will be determined whether the approximate numerical estimates converge to the real solution [3].
By understanding the concepts of zero stability, the research will give a high focus on discussing and studying the convergence properties of Euler's method and its role in approximating solutions to initial value problems via local truncation error and by performing convergence analysis, which will make us evaluate the accuracy and effectiveness of numerical methods in various scientific and engineering applications.
We must take into account the local truncation error (LTE) in order to verify the accuracy and in order to achieve convergence and analysis of Euler's method, because LTE will measure the error that occurs at each time step as a result of derivative approximation, and we can express LTE as follows[4]:

$$\boldsymbol{LTE = O\,(h^2)} \qquad \mathbf{(2)}$$

We find that equation (2) contains the function O(h^2), which indicates that the local cutting error decreases rapidly with a decrease in the step size h. This indicates that the approximation is more accurate by reducing the step size, as Euler's method is a numerical mathematical approximation mechanism. It is approved for solving ordinary differential equations (ODEs). It also focuses in its work on the idea of approximating the derivative of a function via the summation of multiplying the finite difference product by using processes of dividing the time interval into small steps, where Euler's method updates the value of the function in each step on the basis of its derivative and according to the approximation property, it is possible to Euler's method gives errors in the solution where LTE measures the error which determines how the error behaves as the step size decreases and is introduced at each step. We find in the equation containing O(h^2) that the error decreases in a quadratic manner with the reduction of the transition parts, i.e. the step. In practice, we can evaluate the accuracy and convergence of Euler's method through (LTE), which is an analysis of the transition behavior related to the step size [5].
All of this is done by reducing the step size so that the approximation becomes more accurate as the LTE decreases rapidly. The Euler property indicates the importance of choosing the appropriate step size to achieve a balance between accuracy and efficiency. It may not always be the most accurate or numerically stable method for solving differential equations, as in arithmetic We consider that Euler's method is a simple and widely used approximation technique, especially for approximate systems or problems with rapidly changing dynamics. There are also many other numerical methods that provide higher accuracy and stability at the expense of additional computational complexity, such as Runge-Kutta methods, where the The LTE of Euler's method is expressed as O(h^2), which indicates that the local truncation error decreases rapidly as the step size decreases, by working to reduce the step size to the point where the approximation becomes more accurate, but attention must be paid to the comparison between accuracy and computational cost when Step size selection for numerical simulations [6-7].
We always need GE, which is the global error assessment that measures the cumulative error over the

CAHIERS MAGELLANES-NS
Volume 06 Issue 2
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ISSN:1624-1940
DOI 10.6084/m9.figshare.26310310
http://magellanes.com/

entire time period.

$$ge = o\,(h) \qquad (3)$$

The global error decreases with decreasing step size h, as indicated by O(h) in equation (3). This indicates that the accuracy of the method improves with smaller step sizes.

A numerical method is considered convergent if the error tends to zero as the step size approaches zero. Therefore, convergence is a crucial property of numerical methods, and for Euler's method to be convergent, there are two foundations for this, which are [8]:

1. The local truncation error (LTE) should tend to zero when the step size approaches zero.
2. As the step size decreases, the starting values must converge precisely to the correct initial value.

He explained the necessary convergence conditions for Euler's method, and this analysis contributed to a deeper understanding of numerical methods for initial value problems. This research aims to analyze the properties of zero stability and convergence for one-step methods, specifically (Euler's method) in linear problems. Therefore, we worked to present the basic equations of Euler's method through their representation, via, and the truncation error. Local and through global error,

Literary Studies:

The accuracy and reliability of numerical methods in approximating these solutions plays a crucial role in many scientific and engineering applications. Therefore, researchers and scientists alike have been interested in the issue of arriving at numerical methods for solving initial value problems in the constantly evolving field of mathematics, where the solution of many given differential equations is found based on an initial condition.

Among the aspects that determine the quality of numerical methods we find two fundamental concepts: zero-sum stability and convergence. Zero stability refers to the ability of the method to produce accurate results in the presence of small distortions or errors in the initial data. It also ensures that the numerical solution remains finite and does not deviate rapidly from the true solution when the step size becomes zero. Convergence also measures the ability of the method to get close to the exact solution when it decreases

- The authors of this article, "Zerostability and Convergence of Initial Value Problems," [9] A. Olade Adesanya, Dr. Mfon Udoh, and A.M. They created a new hybrid block approach to address third-order initial value problems in ordinary differential equations in general, where this procedure develops a system of nonlinear equations by combining assembly and interpolation of solutions of the power series approximation. A multi-step continuous hybrid linear approach is also obtained by solving this system, and a discrete block method is obtained by extracting the continuous hybrid block method and evaluating it at certain points in the network. Investigations on the properties of the discrete block method confirmed the convergence, coherence and zero stability of these blocks. The proposed technique has been tested using numerical examples, that is, it provides better approximations than existing methods
- The authors of the paper "An Efficient Hybrid Numerical Scheme for Solving Second-Order General Initial Value Problems (IVPs)," [10] B. S. Ogundare and A. O. Adeniran, introduce a new method, namely, zero stability and convergence of initial value problems, which are the

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main topics of the study, which the researchers used in order to construct A hybrid one-step two-site off-grid numerical approach, the scheme exhibited fourth-order accuracy and zero stability. It has also been shown that this approach uses variable step sizes for simultaneous estimation of the approximate solution at both step and step points. Numerical results are presented to demonstrate the improved efficiency of the proposed technique compared to existing methods.

- An innovative point of view is examined in a paper entitled "Boundary value approach to the numerical solution of initial value problems by multi-step methods" [11] by researchers Pierluigi Amodeo and Francesca Mazzia. The researchers' work focused on problems of initial value, instability, and convergence in order to compute the general solution of linear difference equations by Incorporating boundary conditions, as the authors were able to secure a theoretical framework based on the application of multistep linear methods to linear equations, the main stability and convergence aspects of the derived methods are thoroughly investigated and compared with those of classical multistep methods in this work. The authors also introduce new integration formulas which they call extended trapezoidal rules, which share the same stability properties as the trapezoidal rule.

Regardless of the arrangement, these extended trapezoidal rules maintain their stability. Numerical examples are shown to verify the theoretical expectations and to provide initial confidence in possible future applications of boundary value methods

- One-step impulsive numerical techniques are introduced in the research work "Convergence, Consistency and Zero Stability of One-Step Impulsive Numerical Methods" [12] by researcher Gui Lichang since the research focuses on impulsive Runge-Kutta techniques, which is a widely used and advanced form of Runge-Kutta methods. ,The research also proves that through the stable zero ,impulsive approach convergence is guaranteed.

Methodology:

One of the fundamental concepts in numerical analysis used to solve initial value problems is zero stability and convergence, so let us consider a linear initial value problem of the form:

$$y'(t) = y(t),\ y(0) = y_0 \qquad (1)$$

Where: α is a constant and y_0 is the initial condition. The solution to this problem can be given by: y(t) = y_0 * e^(lectt). Now let us solve this problem using Euler's method. Using the formula below, this method updates the solution at each time step:

$$y_{n+1} = y_n + h * f(t_n, y_n)(2)$$

y_n is the approximation of the time function y(t) given by t_n, h is the step size f(t_n, y_n) is the derivative of y(t) evaluated at (t_n, y_n) and Euler's method by applying to equation (1):

$$y_{n+1} = y_n + h * lect * y_n = (1 + h * lect) * y_n(3)$$

The main issue is related to the stability of the numerical solution. Therefore, through this equation, we find that Euler's method introduces the factor (1 + h * lect) within each time step, as this factor plays a decisive role in the stability and convergence of the numerical solution. To find out why, we will analyze the method, where in order for the solution to be The numerical meaning is valid and must remain associated with an increase in the number of time steps or with a decrease in the step size.

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The analysis aims for the numerical solution to approach the real solution with a smaller step size, and thus this is an indication that h is approaching zero, so the numerical approximation converges to an accurate solution.

If the absolute value of the given function (1 + h*lect) is greater than 1 for any time step, it is an explanation because the numerical solution diverges and becomes infinite. Therefore, Euler's method will show zero stability only when (1 + h*lect) is less than or equal to 1 in absolute value for all time steps.

Which leads to inaccurate and unreliable results. This indicates that Euler's method is unstable for certain values of π and step sizes. In order to alleviate this dilemma, which causes many problems, alternative numerical methods such as Runge-Kutta methods can be used.

The Runge-Kutta method provides better stability properties and higher accuracy by considering multiple intermediate steps during each time period. The Euler method presents stability problems, known as zero stability, for the entity of given combinations of π and step sizes. Thus, we will reach divergent numerical solutions and limit the possibility Apply it to solve exact initial value problems.

There are many problems, including the two most complex problems, which are derived from the previously mentioned initial state, which we will work to solve using Euler's method: 1. Second-order linear ODE, taking into account the second-order linear ordinary differential equation:

$$y''(t) + 4y'(t) + 4y(t) = 0, y(0) = 1, y'(0) = 0 \quad (4)$$

We can convert it into a system of first-order equations. In order to solve this equation using Euler's method, we put a variable function, for example z(t), such that z(t) = y'(t), where we will arrive at the following system:

$$y'(t) = z(t)z'(t) = -4z(t) - 4y(t)y(0) = 1\ z(0) = 0$$

Applying Euler's method to In this system, we have:

y_(n+1) = y_n + h * z_n z_(n+1) = z_n + h * (-4z_n - 4y_n)

Starting from initial conditions $y_0 = 1$ and $z_0 = 0$, we can repeat the above equations for different time steps (h) to approximate the numerical solution. 2. Nonlinear ODE: Consider the nonlinear ordinary differential equation:

y'(t) = y(t)^2 + t, y (0) = 0 (5)

To solve this equation using Euler's method, we can transform it back into a system of first-order equations:

u(t) = y(t)^2 u'(t) = 2y(t) * y'(t) = 2y(t) * (y(t)^2 + t) y'(t) = u '(t) y(0) = 0 u(0) = 0

Applying Euler's method to this system, we get:

y_(n+1) = y_n + h * u_n u_(n+1) = u_n + h * (2y_n * (y_n^2 + t_n))

Starting from initial conditions $y_0 = 0$ and $u_0 = 0$, we can iterate these equations for different time steps (h) to obtain an approximate numerical value solution. Euler's method provides an approximate method for solving the given differential equations in both cases. We find that it depends on the accuracy of the numerical solution and its stability on the choice of the step size and the behavior of the original system. We can obtain a series of approximate values that converge in solving the corresponding differential

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equation by repeatedly applying Euler's method.
It may not always produce accurate results for complex or nonlinear systems, but it should be noted that Euler's method and other numerical methods may be more appropriate.
We can extract the local truncation error of Euler's method according to the analysis of the local truncation error by using Taylor series expansion, where we impose sufficient smoothness for y(t) and the local truncation error can be expressed as

$$LTE = \left(\frac{h^2}{2}\right) * y''(t) + O(h^3).$$

The local truncation error is proportional to the square of the step size (h^2) and includes the second derivative of y(t), as this formula shows it. We note that the mathematical function O(h^3) expresses the error in a higher order, as it becomes smaller when h approaches zero.
The local truncation error in each step process will help us measure the accuracy of the approximation within the case of second-order linear ODE (Equation 4) where we can apply Euler's method to approximate the numerical solution by repeatedly calculating y_n+1 using the given formula.
Likewise, in the case of nonlinear ODE (Equation 5), we can use Euler's method to obtain an approximate numerical solution by tracking the local truncation error using the LTE formula where we can evaluate the accuracy of the numerical approximation. It is important to note that although Euler's method provides a simple approach to approximate solutions of differential equations, it may introduce large errors for certain types of problems, especially when the step size is large or the system shows nonlinear behavior
Evaluating consistency and stability: To check the consistency of Euler's method we examine whether the local truncation error decreases as the step size decreases, and in the case of Euler's method the local truncation error is proportional to the square of the step size. So as the step size decreases, the local clipping error should decrease indicating consistency.
To evaluate stability, we need to check whether small perturbations in the initial or algorithmic conditions lead to large errors. In the case of Euler's method, it is known to have certain stability constraints, specifically the method is conditionally stable which means that for some differential equations there is a maximum step size beyond which the numerical solution becomes unstable and errors grow rapidly. For second-order linear ODE and nonlinear ODE the stability behavior of Euler's method must be analyzed. This may include examining the eigenvalues of the coefficient matrix (in the case of linear systems) or performing a stability analysis of the given differential equation. In summary, although Euler's method is consistent, its stability depends on the specific differential equation being solved and the step size chosen.

Conclusions

At the end of the research, we conclude that the concept of zero stability and convergence are the essence of success in applying numerical methods to solve initial value problems. We analyzed the convergence properties of one-step methods, especially Euler’s method, in the case of linear problems. We also studied the local sub-shear error and the general error of Euler’s method, and we also characterized the error. In general, using the maximum local error approximation and time step size, we also looked at some studies that developed in the analysis of other methods for single-step and nonlinear problems.

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310310
http://magellanes.com/

We were able to estimate the local error of Euler's method by, and this indicates that the local error decreases rapidly with decreasing step size. The accuracy and convergence of Euler's method can be evaluated by analyzing the local error and its behavior with respect to the step size.
Using Euler's method to approximate the solution of initial value problems is a simple and common method, but there are other numerical methods such as Runge-Kutta methods, which provide higher accuracy and stability at the expense of the complexity of the calculations, but Euler's problem may not be accurate or numerically stable in all cases, especially in random systems. Or problems with rapidly changing dynamics.
This research aims to highlight the role of Euler in approximating solutions to initial value problems and to analyze the properties of zero stability and convergence. We set the basic equations for Euler's method, including its representation. We noted the local sub-error in addition to the general error. We set the necessary convergence conditions and explained them according to Euler's method.

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