CAHIERS MAGELLANES-NS
Volume 06 Issue 2
2024

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310350
http://magellanes.com/

A DERIVATION OF MULTISTEP IMPLICIT-METHOD WITH THIRD DERIVATIVE FOR SOLVING FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

First-order linear or nonlinear ordinary differential equations (ODEs) can be solved with the help of single-step or multistep numerical methods. This paper discusses multistep numerical methods. With the goal of producing a more efficient multistep numerical method, this work will construct a general implicit method (GI3SM) with a third derivative to directly solve the general category of quasi-linear first-order Ordinary Differential Equations (ODEs), which is expressed as $\psi'(\chi) = \phi(\chi, \psi(\chi))$. The necessary Hermite interpolating polynomials with (GI3SM) have been derived in three steps (IVPs) in order to implement the new efficient multistep numerical technique. A multi-step method is developed for solving this problem, provided that the numerical approximation at three steps is acceptable. To complete the derivation of the multistep implicit method for solving ordinary differential equations of the first order by adding a third derivative. To evaluate the effectiveness of the process, we have looked at four tested examples. The accuracy and efficacy of the suggested method are contrasted to Classical RK and Euler numerical methods with the exact solutions to these problems using the numerical solutions of the implementations. In addition to studying the order and zero-stability of the proposed method, a number that of characteristics have also been established. The implicit multistep proposed method (GI3SM) yields result that are in good agreement with analytical solutions when compared to the classical RK and Euler methods. Additionally, the (GI3SM) generates exact numerical answers for the test problems.

Keyword: Multistep-Method; Implicit-Method; IVPs; ODEs; Three-steps method; Order; RK; First-order; Quasi-linear.

1. Introduction

One of the most significant mathematical tools to recognize physical phenomena is the differential equation (DE). Applied mathematics is used significantly in many domains, such as biology, engineering, physics, economics, chemistry, and medicine. DEs are used to build mathematical models for specific problems in the fields of applied science and engineering. Mathematicians' brains and intelligence is currently being challenged by the difficulty of finding numerical or analytical solutions to some classes of DEs of various forms. Currently, the growth of current numerical approaches is of more interest to mathematicians, physicists, and engineers. Several methods are currently in use to find or examine the solutions of mathematical models with first-order initial value problems (IVPs), which are included below: [1] developed a general implicit-block technique for fifth-order ODEs that uses two points and further derivatives, [2] developed an

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310350
http://magellanes.com/

explicit embedded pair RK method(TFRKF6(5)) to estimate the approximate solution of the oscillatory first-order IVPs to estimate the approximate solution of the oscillatory first-order IVPs while [3] evaluated single-step RK approaches with unique qualities before studying the issues with pronounced oscillatory behavior that frequently occur in quantum and celestial physics. In addition to talking about implicit approaches, stability analysis, error estimation techniques, and dense output. A historical overview of RK methods was provided by [4], who also highlighted the early contributions of Runge, Heun, Kutta, and Nyström, which gave rise to the idea of the order of the accuracy of RK approaches. and [5] investigated at implicit Runge-Kutta processes (IRK). Furthermore, a new explicit RK-technique with two algebraic orders with dispersion and dissipation was introduced by [6]. For the purpose of [7] investigation into the dynamics of a continuous-time system described by an ODE that integrated to obtain trajectories. Lastly, for some problems of various orders, [8,9,10,1,11,12,13] developed multistep numerical algorithms. A brand-new implicit block approach with a three-point second derivative has been given forward in this study. In order to increase the accuracy of approximation solutions to IVPs, (GI3SM) is constructed by adding the first derivative of $\psi(\xi, \eta(\xi))$ using Hermite polynomials. To generate more accurate and comprehensive numerical outputs, the IVP formula employs more derivatives. The applications of several first-order IVP problems are also examined, and the results reveal that the suggested approach has highly good results. As a result, the proposed (GI3SM) technique generates more precise the test problems' numerical solutions as opposed to merely Traditional RK and Euler methods, which are quite compatible with the analytical results. In order to demonstrate the value of the (GI3SM) method, a few test examples have been solved. The numerical outcomes were also as opposed to equivalent numerical outcomes attained with the already-in-use RK and Euler methods. The efficacy and precision of the suggested approach are numerically contrasted alongside the more common Rung-Kutta method and Euler method. To achieve remarkable numerical results for the new three-step methodology, applications of IVPs are also introduced. The numerical test issue solutions obtained by applying the RK and Euler methods closely agree with those obtained by the proposed (GI3SM) method.

2. Quasi-Linear First-Order ODEs

In this section, a quasi-linear, first-order ODEs is studied. The family of general quasi-linear, first-order ODEs is given by the following form:

$$\omega'(\zeta) = \psi\big(\zeta, \omega(\zeta)\big), \qquad \zeta_0 \le \zeta \le \zeta_1, \tag{1}$$

Where,

$$\psi: \Re \times \Re^N \to \Re^N, \omega(\xi) = [\omega_1(\zeta), \omega_2(\zeta), \ldots. , \omega_N(\zeta)]$$

and

$$\psi\big(\zeta, \omega(\zeta)\big) = (\psi_k(\zeta, \omega_i(\zeta))),$$

for $k, i = 1,2, \ldots, N$.

3. Derivation of The Method

For the purpose of solving first-order ODEs, we presented the derivation of the three-step

CAHIERS MAGELLANES-NS
Volume 06 Issue 2
2024

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310350
http://magellanes.com/

implicit multistep approach with third derivatives in this section (IVPs). This work develops a three-step approach for addressing first-order initial value problems with a third derivative (IVPs):

$$\psi'(\zeta) = f\big(\zeta, \psi(\zeta)\big), \psi(a) = \psi_0, \qquad a \le \zeta \le b \tag{2}$$

The second- and third-derivatives for Equation (2) with respect to ζ can be written as

$$\psi''(\zeta) = f'\big(\zeta, \psi(\zeta)\big) = f_\zeta\big(\zeta, \psi(\zeta)\big) + f\big(\zeta, \psi(\zeta)\big) f_\psi\big(\zeta, \psi(\zeta)\big) = g\big(\zeta, \psi(\zeta)\big) \tag{3}$$

$$\begin{aligned}\psi'''(\zeta) &= f''\big(\zeta, \psi(\zeta)\big) \\ &= f_{\zeta\zeta}\big(\zeta, \psi(\zeta)\big) + 2f\big(\zeta, \psi(\zeta)\big) f_\zeta\big(\zeta, \psi(\zeta)\big) + f^2\big(\zeta, \psi(\zeta)\big) f_{\psi\psi}\big(\zeta, \psi(\zeta)\big) \\ &\quad + f_\zeta\big(\zeta, \psi(\zeta)\big) f_\psi\big(\zeta, \psi(\zeta)\big) + f\big(\zeta, \psi(\zeta)\big) f_{\psi\psi}\big(\zeta, \psi(\zeta)\big) \\ &= k\big(\zeta, \psi(\zeta)\big) \qquad (4)\end{aligned}$$

The proposed method is derived from Hermite's interpolating polynomial *P*, which has been defined as follows:

$$P(\zeta) = \sum_{i=0}^{n} \sum_{k=0}^{m_{i-1}} f_i^{(k)} L_{i,k}(\zeta) \tag{5}$$

where $\zeta_i = a + ih$ and $f_i = f(\zeta_i)$ for $i = 0,1, \ldots\ldots, n$ where $h = \frac{b-a}{n}$, and n = positive integer number and $L_{i,k}(\zeta)$ is an arbitrary generalized Lagrange polynomial, $i = 0,1, \ldots., n, k = 0,1, \ldots., m$.

Integrating (2) over the interval $[\zeta_n, \zeta_{n+3}]$ gives:

$$\int_{\zeta_n}^{\zeta_{n+3}} \psi'(\zeta) d\zeta = \int_{\zeta_n}^{\zeta_{n+3}} f(\zeta, \psi(\zeta)) d\zeta \tag{6}$$

$$\psi(\zeta_{n+3}) = \psi(\zeta_n) + \int_{\zeta_n}^{\zeta_{n+3}} f(\zeta, \psi(\zeta)) d\zeta \tag{7}$$

When $f(\zeta, \psi(\zeta))$ in (7) is replaced by the Hermite interpolating polynomial in Reference 2 $dx = hds$, and the integration limit is changed from -3 to 0 in (7), the following results are obtained:

$$\psi(\zeta_{n+3}) = \psi(\zeta_n) + \int_{-3}^{0} \left[\sum_{i=0}^{3} \Big(f_i L_{i,0}(s)\Big) + \sum_{j=0}^{3} (g_j L_{j,1}(s) + k_j L_{j,2}(s)) \right] hds \tag{8}$$

where $i = 0,1,2,3, j = 0,3$ and

$$L_{0,0}(s) = s^3(s+1)(s+2)\left(-\frac{1}{54} - \frac{5(s+3)}{108} - \frac{47(s+3)^2}{648}\right),$$

$$L_{1,0}(s) = (s+1)\left(\frac{s(s+3)}{2}\right)^2,$$

$$L_{2,0}(s) = -(s+2)\left(\frac{s(s+3)}{2}\right)^3,$$

$$L_{3,0}(s) = (s+1)(s+2)(s+3)^3\left(\frac{1}{54} - \frac{5s}{108} + \frac{47s^2}{648}\right),$$

$$L_{0,1}(s) = -hs^3(s+1)(s+2)(s+3)\left(\frac{1}{54} + \frac{5(s+3)}{108}\right),$$

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310350
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$$L_{3,1}(s) = hs(s+1)(s+2)(s+3)^3\left(\frac{1}{54} - \frac{5s}{108}\right),$$

$$L_{0,2}(s) = -\frac{h^2}{108}s^3(s+1)(s+2)(s+3)^3,$$

$$L_{3,2}(s) = \frac{(hs)^2}{108}(s+1)(s+2)(s+3)^3.$$

The integrals in (8) are evaluated. The three-step implicit multistep block method's formula is generated by MAPLE and is as follows:

$$\psi_{n+3} = \psi_n + \frac{h}{1120}(2(1173f_n + 2187f_{n+1} + 2187f_{n+2} + 1173f_{n+3}) + 117h(g_n - g_{n+3}) + 9h^2(k_n + k_{n+3}) \qquad (9)$$

4. The Order of the GI3SM Method

The order of the three-step implicit multistep technique that this paper has derived is established in this section. The local truncation error related to the normalized form of the new technique able to ascertain with the operator for linear difference, as per the research findings of Fatunla [1] and Lambert [2].

$$L[Z(\zeta);h] = \sum_{i=0}^{k}[\alpha_i Z(\zeta+ih) - h\beta_i Z'(\zeta+ih) - h^2\gamma_i Z''(\zeta+ih) - h^3\delta_i Z'''(\zeta+ih)], \qquad (10)$$

If $Z(\zeta)$ is adequately differentiable, the terms in (10) can be expanded as a Taylor series focused on the point ζ to yield the formula.

$L[Z(\zeta);h] = C_0Z(\zeta) + C_1hZ'(\zeta) + \cdots + C_ph^pZ^{(p)}(\zeta)$ where C_p is a constant coefficient and $p = 0,1,\ldots$ are listed below:

$$C_0 = \sum_{l=0}^{k}\alpha_l$$

$$C_1 = \sum_{l=0}^{k}(l\alpha_l - \beta_l)$$

$$C_2 = \sum_{l=0}^{k}\frac{l^2}{2!}\alpha_l - \sum_{l=0}^{k}l\beta_l - \sum_{l=0}^{k}\gamma_l$$

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$$C_p = \frac{1}{p!}\sum_{l=0}^{k}l^p\alpha_l - \frac{1}{(p-1)!}\sum_{l=0}^{k}l^{p-1}\beta_l - \frac{1}{(p-2)!}\sum_{l=0}^{k}l^{p-2}\gamma_l - \frac{1}{(p-3)!}\sum_{l=0}^{k}l^{p-3}\delta_l$$

According to Henrici [3] we say that the proposed method of order p if $C_{p+1} \neq 0$ and $C_k = 0$ where $k = 0,1,2,\ldots,p$. However, the error constant is C_{p+1} and the main local truncation error at the point

ISSN:1624-1940
DOI 10.6084/m9.figshare.26310350
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$x_n = C_{p+1}h^{p+1}Z^{(p+1)}(x_n)$. However, the 3-step implicit multistep method has order p = 10 and error constant $C_{11} = -\frac{27}{68992000}$.

5. Numerical Results

This section involves applying the proposed methods combined with the classical RK and Euler methods to solve a set of quasi-linear first-order ODEs. Three numerical solutions versus exact solutions are compared in Figure 1 to show which is the closest match. Following are some notation examples:

- Step: Step-size used.
- GI3SM: Proposed method.
- RK: Runge-Kutta method.
- Euler: Euler method.

6. Implementations

This section implies solutions to four problems, the numerical outcomes of which are displayed in Figure 1.

Example 6.1. (*Homogenous-Linear ODE*) Consider

$$\varphi'(\tau) = \varphi(\tau) \qquad 0 < \tau \leq 1.$$

with the initial condition (IC): $\varphi(0) = 1$ and the analytical-solution is $\varphi(\tau) = e^{\tau}$.

Example 6.2. (Non-*Homogenous-Linear ODE*) Consider

$$\varphi'(\tau) = \omega(\tau) + \cos(\tau) - \sin(\tau) \qquad 0 < \tau \leq 1.$$

with IC: $\varphi(0) = 0$ and the analytical-solution is $\varphi(\tau) = \sin(\tau)$.

Example 6.3. (Non-*Linear ODE*) Consider

$$\varphi'(\tau) = -\varphi^2(\tau) \qquad 0 < \tau \leq 1.$$

with IC: $\varphi(0) = 1$ and the analytical-solution is $\varphi(\tau) = \frac{1}{1+\tau}$.

Example 6.4. (Non-*Constant Coefficients ODE*) Consider

$$\varphi'(\tau) = -2\tau\varphi(\tau) \qquad 0 < \tau \leq 1.$$

with IC: $\varphi(0) = 1$ and the analytical-solution is $\varphi(\tau) = e^{-\tau^2}$.

CAHIERS MAGELLANES-NS
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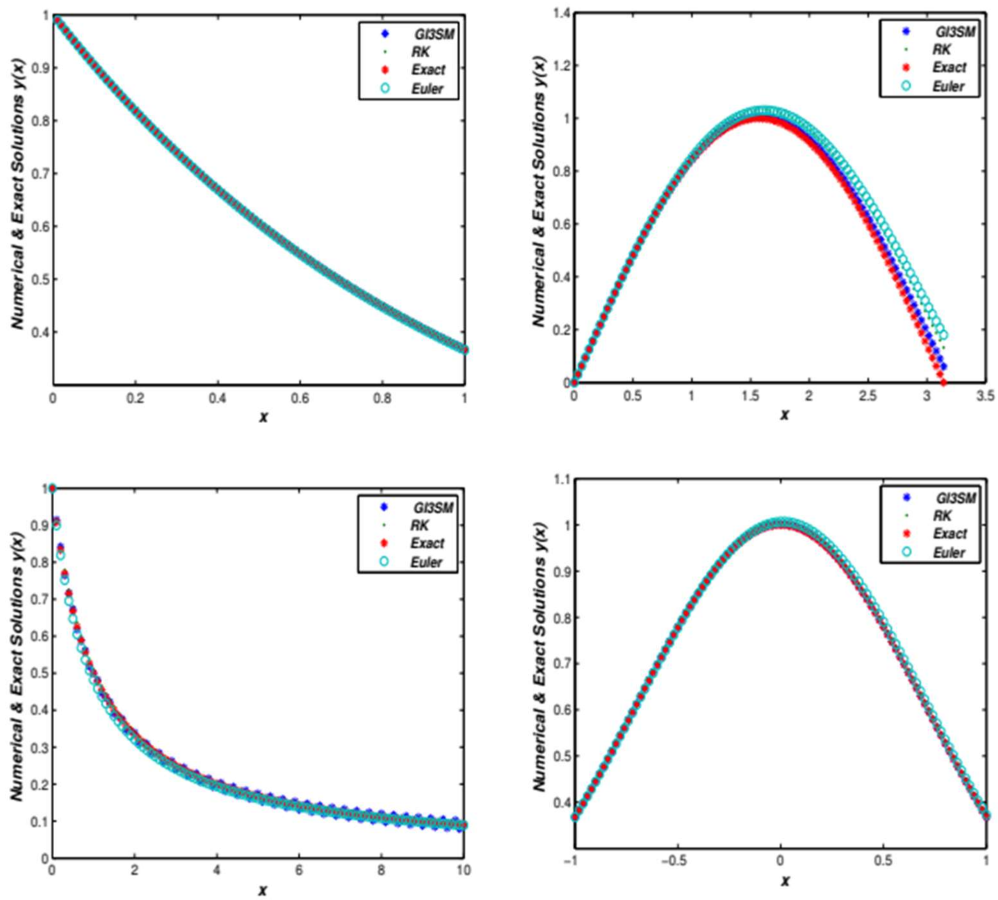

Figure 1: A Comparison of Numerical Solutions of Proposed Method Versus Numerical Solutions of Classical RK and Euler Methods with Exact Solution for the Examples 1,2,3 and 4.

7. Conclusion and Discussion

The general implicit block method (GI3SM) has been developed in this study to solve the general class of quasi-linear first-order ODEs using the Hermite approximation technique. The approach is called (GI3SM). For the general class of first-order ODEs, this article's objective is to demonstrate a direct-implicit block solution approach. Comparisons between the numerical solutions produced by the proposed (GI3SM) technique and those produced by the (GI3SM) method for the same order have been made. By comparing the two approaches, we can say that the new (GI3SM) method surpasses the classical RK and Euler methods across a range of advantages. The implementation's findings enable us to come to the conclusion that the proposed method is a good one for computation, required less function evaluation and is more time- and cost-effective.

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