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HYERS-ULAM-RASSIAS STABILITY OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAY

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Abstract
The intention of this article is to analyze the Hyers-Ulam-Rassias Stability (HURS) of differential equations of second order with state dependent delay.

Keywords: Stability, Hyers Ulam Rassias Stability, Differential equations, Dependent delay.

2020 Mathematics Subject Classification:34K43, 93C10, 93C55

1. Introduction

Hernendoz and et al [1] ascertained the state- dependent delay condition by employing first order partial differential equations and abstract differential equations in the study publications of [2]. Subsequently, a scrutiny of the fore stated research and contemporary endeavors provided by Kristin et .al [3] has been focused. A variation application of the calculus of variation is the configuration of the differential equation of second order system. The research of state-dependent delays involving partial differential equations of second order and abstract differential equation, has been effectively exposed through the review of literature for references, see $[4-6]$.

The correlations between the disquisition of the cited articles [7-11] and the illustration of differential equations enumerated in definite spatial position by Aiello, Freedman and Wu [12] are additionally ascertained. Inspite of that S.M.Ulam defied the problem of stable functional equations in 1940. A multitude of mathematicians [22-23] have made an intensive exploration on the stability problems with functional equations, since the introduction and the identification of the problem concerned.

In 1978, M.Rassias besowed the Generalized Hyer-Ulam Stability (also termed as Hyers-Ulam-Rassias Stability-HURS) of this Hyers Ulam Stability [24]. Consequently, numerous researchers initiated their research on the Hyers-Ulam Stability (HUS) of functional equation and differential equations, which ended up in the incredible and in dispensable progress, which is currently witnessed. As a result, ample numbers of authors [25-34] and even more have successfully carried out extensive research on the HUS.

Occasionally termed as the HURS of first order, higher order, and fractional order differential equations. Correspondingly, the research on the HUS has attained a prominent position in the arena of mathematics.

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In 2018, E. Hernández et.al. [13] proved the existence and uniqueness of solution for the following problem.

$$y''(\tau) = \mathfrak{A}y(\tau) + \mathcal{F}\big(\tau, y_{\rho(\tau,y_\tau)}\big),\ \tau \in [0,\mathfrak{b}] \tag{1}$$
$$y_0 = \beta \in \mathcal{B} = C([-\gamma,0];X), y'(0^+) = x \in X \tag{2}$$

Let $(X, |\cdot|)$ be a Banach space. Define $B: U \to X$ is a bounded and linear operator, here U is a Banach space and denotes the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(\mathcal{C}(\tau))_{\tau\in\mathbb{R}}$ on $(X, |\cdot|)$ and $\mathcal{F}(\cdot), \rho(\cdot)$ are appropriate functions; the function $y_\tau: (-\infty, 0] \to X, y_\tau(\theta) = y(\tau+\theta)$, is a member of particular abstract state space $\mathcal{B}$ termed clearly; $0 < \tau_1 < \cdots < \tau_n < \mathfrak{b}$ are annexed numbers; $\rho: \mathcal{J} \times \mathcal{B} \to (-\infty, \mathfrak{b}]$ is a suitable function.

The stat-dependent delay and stability of the abstract differential system is considered as the captivating and compelling concept of current investigation. The pre-illuminant purpose of this research is to successfully analyze the stability of the foretasted problem by applying HUS and HURS.

2. Preliminaries

Let $(\mathcal{V}, |\cdot|_{\mathcal{V}})$ and $(\mathcal{W}, |\cdot|_{\mathfrak{W}})$ represents Banach spaces. Let $|\cdot|_{\mathcal{L}(\mathcal{V},\mathfrak{W})}$ denotes a space of linear and bounded operator norm function, $\mathcal{L}(\mathfrak{V},\mathfrak{W}): \mathfrak{v} \to \mathfrak{W}$.

The space $\mathfrak{E} = \{§ \in X: \mathcal{C}(\cdot)§$ is continuosly differentiable $\}$ awarded into the norm $|x|_{\mathfrak{E}} = |§| + \sup_{0\leq\tau\leq b}|\mathcal{AS}(\tau)§|$. From the literature of Kisiński [21], since $\mathfrak{E}$ is a Banach space, $\mathcal{AS}(\tau) \in \mathcal{L}(\mathfrak{E}, X)$ for every τ in the real line $\mathbb{R}$ and if s converges to 0 then $\mathcal{AS}(s)§$ converges to 0, for every $§ \in \mathbb{E}$.

Using the references from the articles, we can comprehend the abstract Cauchy problem of second order and cosine functions better [21]. Let $C([p,q];X)$ and $C_{Lip}([p,q];\mathcal{V})$ be normally defined spaces and its norms are denoted as $|\cdot|_{C([p,q];X)}$ and $|\cdot|_{C_{Lip}([p,q];X)}$ respectively. We termed that

$$|\cdot|_{C_{Lip}([p,q];X)} = |\cdot|_{C([p,q];\mathcal{U})} + [\cdot]_{C_{Lip}([p,q];\mathfrak{U})}$$

where $[\xi]_{C_{Lip}([p,q];\mathcal{V})} = \sup_{\tau,s\in[p,q],s\neq\tau} \frac{|\xi(s)-\xi(\tau)|_{\mathcal{V}}}{|\tau-s|}$ and using the state-dependent delay from [16].

Lemma 2.1. [16, Lemma 1]

Suppose $y, z \in C([-\alpha, a]; X), 0 < a \leq b$, the function $\rho(\cdot)$ belongs to $C_{\text{Lip}}([0,b] \times \mathcal{B}; \mathbb{R}^+), y_{[0,a]}, z_{[0,a]} \in C_{\text{Lip}}([0,a];X), y_0 = z_0 = \vartheta \in C_{\text{Lip}}([-\alpha,0];X)$, and $\rho(\tau, h_\tau) \leq$ for $\eta = y, z$ and every $\tau \in [0,a]$.

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$$\left[y_{\rho(\cdot,y(\cdot))}\right]_{C_{Lip}([0,a];\mathcal{B})} \leq \left[y_{(\cdot)}\right]_{C_{Lip}([0,a];\mathcal{B})}[\rho]_{C_{Lip}([0,a]\times\mathcal{B};\mathbb{R}^+)}\left(1+\left[y_{(\cdot)}\right]_{C_{Lip}([0,a];\mathcal{B})}\right) \quad (3)$$

$$\left|y_{\sigma(\cdot,y_{(\cdot)})} - z_{\sigma(\cdot,z(\cdot)}\right|_{C([0,a];\mathcal{B})} \quad (4)$$

$$\leq \left(1+\left[z_{(\cdot)}\right]_{C_{Lip}([0,a];\mathcal{B})}[\rho]_{C_{Lip}([0,a]\times\mathcal{B};\mathbb{R}^+)}\right)|y-z|_{C([0,a];X)} \quad (5)$$

The presence and originality of the answer are first investigated. The validity of the problem is then shown (1)-(2). First, we outline the modest and conventional approach to (1)-(2).

Definition 2.1. [13]

The mild solution function $y \in C([0,a];X)$ of (1)-(2) on $[-\alpha,a]$, when $0 < a \leq b$, is defined by

$$y(\tau) = \mathcal{C}(\tau)\beta(0) + \mathcal{S}(\tau)x + \int_0^\tau \mathcal{S}(\tau-s)\mathcal{G}\left(s, y_{\rho(s,y_s)}\right)ds$$

for all $\tau \in [0,a]$ and if $y_0 = \beta$.

Now, we may acquire the initial primary result.

Theorem 2.1. [6]

Let (X,d) be a generalised complete metric space. Assume that $\Phi: X \to X$ is a lipschitz-constant strictly contractive operator $L < 1$, If there exists a nonnegative integer k such that $d(\Phi^{k+1}x, \Phi^k) < \infty$ for some $x \in X$, then the following are true:

(a) The sequence $\{\Phi^n x\}$ converges to a fixed end point x^* of Φ.

(b) x^* is the unique fixed point of Φ in $X^* = \{y \in X / d(\Phi^k x, y) < \infty\}$.

(c) If $y \in X^*$, then $d(y,x^*) \leq \frac{1}{1-L} d(\Phi y, y)$.

3. Hyers-Ulam-Rassias Stability and Hyers-Ulam Stability

This section's primary objective is to investigate the Hyers-Ulam-Rassias stability (H-U-R-S) and Hyers-Ulam stability of the second order differential equation (H-U-S) (1)-(2).

Theorem 3.1. [13]

Let us assume that $\rho \in C_{\text{Lip}}([0,b]\times\mathcal{B};\mathbb{R}^+), \rho(0,v) = 0$ and a non-negative number r^* exists and $0 < a^* \leq b$ provides $0 \leq \rho(\tau,\vartheta) \leq \tau$ for every $\tau \in [0,a^*]$ and $\vartheta \in B_{r^*}(\beta,\mathcal{B})$. Also, $\beta \in C_{\text{Lip}}([-\alpha,0];X), \mathcal{C}(\cdot)\beta(0) \in C_{\text{Lip}}([0,a];X)$ and $\mathcal{G} \in C_{\text{Lip}}([0,b]\times\mathcal{B};X)$ holds. Then exactly one mild solution $y \in C_{\text{Lip}}([-\alpha,a];X)$ of problem (1)-(2) on $[-\alpha,a]$ for any $0 < a \leq b$, exists.

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Proof:

Let d^* and r^* be defined in condition $\mathbf{H}_{\rho,v}$. Let $\mathcal{R} > 0$ be sufficient and it satisfies that $\mathcal{R} > [\beta]_{C_{Lip}([-\alpha,0];X)} + [\mathcal{C}(\cdot)v(0)]_{C_{Lip}([0,b];X)} + C_0|x|$. Let us consider $\mathcal{R}a \le r^*$ when $0 < a < \min\{b, d^*, 1\}$ and

$$C_0 a^2 L_g \left(1 + \mathcal{R}[\rho]_{C_{Lip}([0,b]\times\mathcal{B};\mathbb{R}^+)} < 1, \tag{6}$$

$$[\mathcal{C}(\cdot)\beta(0)]_{C_{Lip}([0,a];X)} + C_0|x| + 2aC_0\left(L_g r^* + \sup_{\tau\in[0,a]}|\mathcal{G}(\tau,\beta)|\right) \le \mathcal{R}. \tag{7}$$

$$\mathcal{G}y(\tau) = \mathcal{C}(\tau)\beta(0) + \mathcal{S}(\tau)x + \int_0^\tau \mathcal{S}(\tau - s)\mathcal{G}(s, y_{\rho(s,y_s)})ds. \tag{8}$$

Let $y \in \mathcal{Y}(a,\mathcal{R})$. By the Lemma 2.1 and the selection of $\mathcal{R}$, we get,

$$|\mathcal{G}(s, y_{\rho(s,y_s)})| \le |\mathcal{G}(s, y_{\rho(s,y_s)}) - \mathcal{G}(s,\beta)| + |\mathcal{G}(s,\beta)|$$
$$|\mathcal{G}(s, y_{\rho(s,y_s)})| \le L_g|y_{\rho(s,y_s)} - \beta| + \sup_{\tau\in[0,a]}|\mathcal{G}(\tau,\beta)|$$

Using (8), for all $\tau \in [0,a)$ and $\tau + \eta \in [0,a]$, for $\eta > 0$,

$$|\mathcal{G}y(\tau+\eta) - \mathcal{G}y(\tau)| \le [\mathcal{E}(\cdot)\beta(0)]_{C_{Lip}([0,a];X)}\eta + C_0|x|\eta + C_0\eta \sup_{s\in[0,a]}|\mathcal{G}(s, y_{\rho(s,y_s)})|a$$
$$|\mathcal{G}y(\tau+\eta) - \mathcal{G}y(\tau)| + C_0 a \sup_{s\in[0,a]}|\mathcal{G}(s, y_{\rho(s,y_s)})|\eta$$

this indicates that $[\mathcal{E}y]_{C_{Lip}([0,a];X)} \le \mathcal{R}$. Furthermore, we know that $(\mathcal{G}y)_0 = \beta, \beta \in C_{Lip}([-\gamma,0];X)$ and $\mathcal{R} > [\beta]_{C_{Lip}([-\gamma,0];X)}$, from Lemma 2.1 we obtain that $\mathcal{G}y \in C_{\mathrm{Lip}}([-\gamma,\mathfrak{a}];X)$ and $[\mathcal{G}y]_{C_{Lip}([-\gamma,\mathfrak{a}];X)} \le \mathcal{R}$, which shows that $\mathcal{G}$ is a $\mathcal{Y}(\mathfrak{a},\mathcal{R})$-valued function.

Contrarily, the Lemma's conclusion was achieved 2.1 and for every $y, z \in y(a,\mathcal{R})$ and $\tau \in [0,a]$ gives,

$$|\mathcal{G}y(\tau) - \mathcal{G}_z(\tau)| \le \int_0^\tau C_0 a L_g |y_{\rho(s,y_s)} - z_{\rho(s,z_s)}|_{\mathcal{B}} ds$$
$$|\mathcal{G}y(\tau) - \mathcal{G}_z(\tau)| \le C_0 a^2 L_g \left(1 + \mathcal{R}[\rho]_{C_{Lip}([0,b]\times\mathcal{B};\mathbb{R}^+)}\right) d(y,z)$$

which exhibits the contraction map $\mathcal{G}$, which denotes the unique solution to (1)-(2) on the interval $[0,a]$ such that $y \in C_{Lip}([-\alpha,a];X)$.

Theorem 3.2 (Hyers-Ulam-Rassias Stability(H-U-R-S)).

Consider the closed interval $I_R = [0,b]$. Let κ_1, C_0 and L_g be non-negative constants with the condition

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$0 < \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip}([0,b] \times B; R^+)\right) < 1.$

Proof:

Assume that the function $F: [0,b] \times \Re \to \Re$ is continuous and satisfying the Lipschitz condition

$$|F(\tau, y_\sigma) - F(\tau, z_\sigma)| \le L_g\left(1 + [z]_{\text{Lip }([-\alpha,a];X)}[\sigma]_{\text{Lip }([0,a]\times B;R)}\right)|y - z|_{G([-\alpha;a],X)} \tag{9}$$

for each $\tau \in I_R$ and $y, z \in \Re$. If $F: [0,b] \to \Re$ is a continuously differentiable function satisfies

$$\left|y'' - Ay(\tau) - F\left(\tau, y_{\sigma(\tau,y_\tau)}\right)\right| \le \vartheta(\tau) \tag{10}$$

$\forall\tau \in [0,b]$, and $\vartheta: [0,b] \to (0,\infty)$ is a continuous function with the inequality

$$\left|\int_0^\tau (\tau - t_1)\vartheta(\tau_1)dt_1\right| \le \kappa_1\vartheta(\tau) \tag{11}$$

$\forall\tau \in [0,b]$, then there is a unique continuous function $y_0: [0,b] \to \Re$ such that

$$y_0(\tau) = C(\tau)\beta(0) + \boldsymbol{\delta}(\tau) + \int_0^\tau \delta(\tau - s)F\left(s, y_{\sigma(s,y_s)}\right)ds \tag{12}$$

and

$$|y(\tau) - y_0(\tau)| \le \frac{\kappa_1}{1 - \kappa_1 C_0 a^2 L_g\left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right)}\vartheta(\tau)\ \forall\tau \in I_R \tag{13}$$

Proof: Define a set

$$\Phi = \{y: [0,b] \to [-\alpha, b] \mid y \text{ is continuous }\} \tag{14}$$

equipped with generalized complete metric

$$d(y,z) = \inf\left\{C \in [0,\infty] \mid |y(\tau) - z(\tau)| \le C_{yz}\vartheta(\tau),\ \forall\tau \in [0,b]\right\} \tag{15}$$

Define an operator $\Psi: \Phi \to \Phi$ by

$$(\Psi y)(\tau) = C(\tau)\beta(0) + \delta(\tau) + \int_0^\tau \delta(\tau - s)F\left(s, y_{\sigma(s,y_s)}\right)ds \tag{16}$$

for each $y \in \Phi$.

We noticed that Ψ is well defined because both F and y are continuous functions.

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In order to reach our goal, we must first prove that Ψ is strictly contractive operator on Φ.

For any $y, z \in \Phi$, assume $C_{yz} \in [0, \infty]$ is arbitrary constant such that $d(y,z) \leq C_{yz}$.

(i.e) from (15) we have

$$|y(\tau) - z(\tau)| \leq C_{yz}\vartheta(\tau) \tag{17}$$

$\forall \tau \in [0, b]$.

From the conditions (9), (11), (14), (15) and (17), we have

$$|(\Psi y)t - (\Psi z)t| \leq \int_0^\tau C_0 a L_g \left|y_{\sigma(s,y_s)} - z_{\sigma(s,z_s)}\right| ds$$
$$(\Psi y)t - (\Psi z)t| \leq \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right) C_{yz}\vartheta(\tau)$$

for all $\tau \in [0, b]$. That is

$$d(\Psi y, \Psi z) \leq \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right) C_{yz}\vartheta(\tau)$$

Hence we can conclude that

$$d(\Psi y, \Psi z) \leq \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right) d(y,z)$$

for any $y, z \in \Phi$.

From (15) and (17), it follows that for any arbitrary $g_0 \in \Phi$, there is a constant C such that $0 < C < \infty$ with

$$|(\Psi y_0)(\tau) - y_0(\tau)| = \left|C(\tau)\beta(0) + \delta(\tau) + \int_0^\tau \delta(\tau - s) F\left(s, y_{\sigma(s,y_s)}\right) ds - y_0(\tau)\right|$$
$$|(\Psi y_0)(\tau) - y_0(\tau)| \leq C\vartheta(\tau)$$

$\forall \tau \in [0, b]$, since $F(\tau, y_0(\tau))$ and $y_0(\tau)$ are bounded in the interval $[0, b]$ and $\min_{\tau\in[0,b]}\vartheta(\tau) > 0$.

Thus (16) it means that

$$d(\Psi y_0, y_0) < \infty$$

As a result, by theorem (2.1), there is a continuous function $f_0: [0, b] \to \Re$ so that $\Psi^n y_0 \to f_0$ in (Φ, d) and $\Psi f_0 = f_0$. This means that, f_0 corresponds to the equation (13) for each $t \in [0, b]$.

Now we validate that $\{y \in \Phi / d(y_0, y) < \infty\} = \Phi$.

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For any value $h \in \Phi$, since h and h_0 are bounded on the interval $[0, b]$ and $\min_{\tau\in[0,b]} \beta(\tau) > 0$, there is a constant $0 < C_h < \infty$ such that $|h_0(\tau) - h(\tau)| \leq C_h \vartheta(\tau)$.

Hence, we must have $d(h_0, h) < \infty, \forall h \in \Phi$. (i.e) $\{h \in \Phi / d(h_0, h) < \infty\} = \Phi$.

Hence in sight of theorem (2.1), we can conclude that f_0 is the unique continuous function with the property (13).

Also, it follows from (10)

$$-\vartheta(\tau) \leq y'' - Ay(\tau) - F\left(\tau, y_{\sigma(\tau, y_\tau)}\right) \leq \vartheta(\tau)$$

$\forall \tau \in [0, b]$.

If we integrate every terms of previous inequality from 0 to, we get

$$-\int_0^\tau \vartheta(\tau) d\tau \leq \int_0^\tau [y''(\tau) - Ay(\tau) - g(\tau, y_\sigma)] d\tau \leq \int_0^\tau \vartheta(\tau) d\tau$$
$$-\int_0^\tau \vartheta(\tau) d\tau \leq y'(\tau) - x(\tau) - \int_0^\tau Ay(\tau) d\tau - \int_0^\tau g(\tau, y_\sigma) d\tau \leq \int_0^\tau \vartheta(\tau) d\tau$$

Again integrating from o to s we get,

$$-\int_0^s \int_0^\tau \vartheta(\tau) d\tau ds \leq y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \leq \int_0^s \int_0^\tau \vartheta(\tau) d\tau ds$$

Now applying the replacement lemma, we obtain

$$-\int_0^\tau (\tau - s)\vartheta(\tau) d\tau \leq y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \leq \int_0^\tau (\tau - s)\vartheta(\tau) d\tau$$
$$\left| y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \right| \leq \left| \int_0^\tau (\tau - s)\vartheta(\tau) d\tau \right|$$
$$\left| y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g\left(s, y_s\right) ds \right| \leq \kappa_1 \vartheta(\tau)$$

$\forall \tau \in [0, b]$.

Hence by conditions (12) and (17) we get

$$|y(\tau) - (\Psi y)(\tau)| \leq \kappa_1 \vartheta(\tau)$$

$\forall \tau \in [0, b]$, which gives

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$$d(y, \Psi y) \leq \kappa_1 \vartheta(\tau) \tag{18}$$

Finally the theorem (2.1) with (18) means that

$$d(y, y_0) \leq \frac{1}{1 - \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{\mathrm{Lip}\,([0,b]\times B; R^+)}\right)} d(y, \Psi y)$$
$$d(y, y_0) \leq \frac{\kappa_1}{1 - \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B; R^+)}\right)} \vartheta(\tau)$$

which completes the proof.

Theorem 3.3 (Hyers-Ulam Stability(H-U-S)).

Consider the closed interval $I_R = [0, b]$. Let κ_1, C_0, and L_g be non-negative constants with the condition $0 < \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{\mathrm{Lip}\,([0,b]\times B; R^+)}\right) < 1$. Assume that the function $F: [0, b] \times \Re \to \Re$ is continuous and satisfying the Lipschitz condition

$$|F(\tau, y_\sigma) - F(\tau, z_\sigma)| \leq L_g \left(1 + [z]_{\mathrm{Lip}\,([-\alpha,a]; X)} [\sigma]_{\mathrm{Lip}\,([0,a]\times B; R)}\right) |y - z|_{G([-\alpha;a], X)} \tag{19}$$

for each $\tau \in I_R$ and $u, v \in \Re$. If $F: [0, b] \to \Re$ is a continuously differentiable function satisfies

$$| y'' - Ay(\tau) - F\left(\tau, y_{\sigma(\tau, y_\tau)} \right| \leq \varepsilon \tag{20}$$

$\forall \tau \in [0, b]$, then there is a unique continuous function $f_0: [0, b] \to \Re$ such that

$$y_0(\tau) = C(\tau)\beta(0) + \delta(\tau) + \int_0^\tau \delta(\tau - s) F\left(s, y_{\sigma(s, y_s)}\right) ds \tag{21}$$

and

$$|y(\tau) - y_0(\tau)| \leq \frac{ts}{1 - \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B; R^+)}\right)} \varepsilon \tag{22}$$

Proof:

Define a set

$$\Phi = \{ y: [0, b] \to [-\alpha, b] \mid y \text{ is continuous } \} \tag{23}$$

equipped with generalized complete metric

$$d(y, z) = \inf\{ C \in [0, \infty] | \; |y(\tau) - z(\tau)| \leq C, \; \forall \tau \in [0, b] \} \tag{24}$$

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Define an operator $\Psi: \Phi \rightarrow \Phi$ by

$$(\Psi y)(\tau) = C(\tau)\beta(0) + \delta(\tau) + \int_0^\tau \delta(\tau - s)F\left(s, y_{\sigma(s,y_s)}\right)ds \tag{25}$$

for each $y \in \Phi$.

To reach our goal, we must first prove that Ψ is strictly contractive operator on Φ.

For any $y, z \in \Phi$, assume $C_{yz} \in [0, \infty]$ is arbitrary constant such that $d(y, z) \leq C_{yz}$.

(i.e) from (15) we have

$$|y(\tau) - z(\tau)| \leq C_{yz} \tag{26}$$

$\forall \tau \in [0, b]$.

From the conditions (19), (25) and (26) we have

$$|(\Psi y)\tau - (\Psi z)\tau| \leq \int_0^\tau C_0 a L_g \left|y_{\sigma(s,y_s)} - z_{\sigma(s,z_s)}\right| ds$$
$$|(\Psi y)\tau - (\Psi z)\tau| \leq \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right) C_{yz}$$

for all $\tau \in [0, b]$. That is

$$d(\Psi y, \Psi z) \leq \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right) C_{yz}$$

Hence, we can conclude that

$$d(\Psi y, \Psi z) \leq \kappa_1 C_0 a^2 L_g \left(1 + \Re[\sigma]_{Lip([0,b]\times B;R^+)}\right) d(y, z)$$

for any $y, z \in \Phi$.

Analogously to the proof of the above theorem, we can show that each $y_0 \in \Phi$ satisfies the property $d(\Psi y_0, y_0) < \infty$.

Therefore, theorem 2.1 implies that there exists a continuous function $y_0: [0, b] \rightarrow \Re$ such that $\Psi^n y_0 \rightarrow y_0$ in (Φ, d) as $n \rightarrow \infty$ and such that $y_0 = \Phi y_0$, that is, y_0 satisfies (12) for any $\tau \in I_R$. From (14) and (16), it follows that for any arbitrary $g_0 \in \Phi$, there is a constant C such that $0 < C < \infty$ with

$$|(\Psi y_0)(\tau) - y_0(\tau)| = \left|C(\tau)\beta(0) + \delta(\tau) + \int_0^\tau \delta(\tau - s)F\left(s, y_{\sigma(s,y_s)}\right)ds - y_0(\tau)\right|$$
$$|(\Psi y_0)(\tau) - y_0(\tau)| \leq C$$

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$\forall \tau \in [0, b]$. Thus (24) it means that

$$d(\Psi y_0, y_0) < \infty$$

Now we validate that $\{y \in \Phi / d(y_0, y) < \infty\} = \Phi$.

For any value $h \in \Phi$, since h and h_0 are bounded on the interval $[0, b]$ and $\min_{\tau \in [0,b]} \beta(\tau) > 0$, there is a constant $0 < C_{f_1 f_2} < \infty$ such that $|h_0(\tau) - h(\tau)| \leq C_h \varepsilon$.

Hence, we must have $d(h_0, h) < \infty, \forall h \in \Phi$. (i.e) $\{h \in \Phi / d(h_0, h) < \infty\} = \Phi$.

Hence in sight of theorem (2.1), we can conclude that f_0 is the unique continuous function with the property (21). Also, it follows from (20)

$$-\varepsilon \leq y'' - Ay(\tau) - F\big(\tau, y_{\sigma(\tau, y_\tau)} \leq \varepsilon$$

$\forall \tau \in [0, b]$.

If we integrate every terms of previous inequality from 0 to , we get

$$-\int_0^\tau \varepsilon d\tau \leq \int_0^\tau [y''(\tau) - Ay(\tau) - g(\tau, y_\sigma)] d\tau \leq \int_0^\tau \varepsilon d\tau \quad (27)$$

$$-\int_0^\tau \varepsilon d\tau \leq y'(\tau) - x(\tau) - \int_0^\tau Ay(\tau) d\tau - \int_0^\tau g(\tau, y_\sigma) d\tau \leq \int_0^\tau \varepsilon d\tau \quad (28)$$

Again integrating from o to s we get,

$$-\int_0^s \int_0^\tau \varepsilon d\tau ds \leq y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \leq \int_0^s \int_0^\tau \varepsilon d\tau ds \quad (29)$$

Now applying the replacement lemma, we obtain

$$-\varepsilon\tau s \leq y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \leq \varepsilon\tau s \quad (30)$$

$$\left| y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \right| \leq |\varepsilon\tau s| \quad (31)$$

$$\left| y(\tau) - \beta(0)c(\tau) - s(\tau)x - \int_0^\tau S(\tau - s) g(s, y_s) ds \right| \leq \varepsilon\tau s \quad (32)$$

$\forall \tau \in [0, b]$.

Hence by conditions (12) and (17) we get

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$$|y(\tau) - (\Psi y)(\tau)| \leq \varepsilon\tau s$$

$\forall\tau \in [0, b]$, which gives

$$d(y, \Psi y) \leq \varepsilon\tau s \tag{33}$$

Finally, the theorem (2.1) with (33) means that

$$d(y, y_0) \leq \frac{1}{1 - \kappa_1 C_0 a^2 L_g\left(1 + \Re[\sigma]_{\mathrm{Lip}\,([0,b]\times B; R^+)}\right)} d(y, \Psi y)$$
$$d(y, y_0) \leq \frac{\tau s}{1 - \kappa_1 C_0 a^2 L_g\left(1 + \Re[\sigma]_{Lip([0,b]\times B; R^+)}\right)} \varepsilon$$

which completes the proof.

4. Conclusion

Hernendoz has proved uniqueness and existence of solutions in second order differential equations with state dependent equations with state dependent delay. For the current research, a fore stated problem is taken into consideration with the inclusion and implementation of HURS. Thus, with the application of HURS the research successfully satisfied the purpose and effectively proved.

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