CAHIERS MAGELLANES-NS
Volume 06 Issue 1
2024

ISSN:1624-1940
DOI 10.6084/m9.figshare.26312694
http://magellanes.com/

INVERSE DOMINATION OF AN UNDIRECTED GRAPH ON A FINITESUBSET OF NATURAL NUMBERS

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Abstract: An undirected graph G_n on a finite subset of natural numbers is an undirected graph whose vertex set is $V = \{x \in N: \gcd(x, n) \neq 1, x < n\}$ and $x, y \in V$ are adjacent if and only if $\gcd(x, y) > 1$. In this paper the concept of inverse domination and inverse total domination is discussed and the results are presented at various n values of G_n.

Keywords: Undirected graph, Domination, Inverse Domination, Inverse total domination.

Mathematics Subject Classification: 05C25, 05C69

1. INTRODUCTION

Domination is the most popular concept in graph theory and it was first introduced by Berge [2] and Ore[12]. The dominating set $D \subset V$ of a graph G (V, E) is defined as every vertex v in $V - D$ is adjacent to some vertex in D and the minimum cardinality of a dominating set is the domination number of G, denoted by $\gamma(G)$. Total domination was introduced by Cockayne et al. [4,5,6] and some concepts of this domination, found in Haynes et al. [8]. The relation between domination number and total domination number of a graph without isolated vertices was given in Bollobas [3]. The concept of inverse domination was given in Kulli et al. [10]. The inverse domination of bipartite and chordal graphs has been studied in [1]. Some relationship between inverse domination of Jump graphs and other parameters was investigated by Karthikeyan et al. [9]. Syeda Asma Kaucer et. Al [13] discussed inverse domination and inverse total domination of an undirected graph $G_{m,n}$ and presented some results for different cases of m, n. In this paper, some results on dominating sets of an undirected graph on a finite subset of natural numbers are presented and the domination numbers are obtained.

2. UNDIRECTED G_n GRAPH AND ITS PROPERTIES

Cayley in 1978 constructed the digraph of a group, thus paving way for many more emerging graphs for semi groups such as divisibility graphs, power graphs, annihilator graphs and so on.

Chakrabarty [7] developed the concept of an undirected graph on a finite subset of naturalmber and it is denoted by G_n.

ISSN:1624-1940
DOI 10.6084/m9.figshare.26312694
http://magellanes.com/

Definition: Let $n \in N$ be a composite number. An undirected G_n graph is defined as a graph whose vertex set $V = \{x \in N: \gcd(x, n) \neq 1, x < n\}$ where $x, y \in$ V are adjacent if and only if $\gcd(x, y) > 1$.

Some of the properties of undirected G_n graph given by Ivy [10] are

Lemma 2.1 [7]: The graph G_n is disconnected if and only if $n = 2p$, where p is an odd prime. Moreover, the components of $G_{2p}$are K_{p-1} and K_1.

Lemma 2.2 [7]: The graph G_n is complete if and only if $n = p^m$, where p is a prime.

3. INVERSE DOMINATION OF A GRAPH G_n

The concept of inverse domination was given in Kulli et al. [10]. The inverse domination of bipartite and chordal graphs has been studied in [1]. Some relationship between inverse domination of Jump graphs and other parameters was investigated by Karthikeyan et al. [9]. In this section results on inverse domination of an undirected graph G_n are discussed and inverse domination number of G_n are obtained for different values of n.

Definition: Let $G(V, E)$ be a graph. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G then D' is called an inverse dominating set with respect to D. The order of the smallest inverse dominating set is called its inverse domination number and it was denotedby $\gamma^{-1}(G)$.

Theorem 3.1: For a graph G_n, the inverse domination number $\gamma^{-1}(G_n) = 1$, if $n = p^m$, where p is prime.

Proof: Consider the graph G_n, where $n = p^m$, and p is prime. Then the vertex set of graph G_n consisting of all multiples of p which are less than n. That means $V = \{p, 2p, 3p, ..., (p^{m-1} - 1)p\}$ and $|V| = p^{m-1} - 1$.

By the definition of graph G_n, it forms a complete graph and degree of each vertex is $|V| - 1$, which is maximum. Therefore every vertex $v_i = i$ in V is adjacent to all $v_j = j$ where $i \neq j$ of G_n as $\gcd(i, j) > 1$. Then $D = \{p\}$ forms a dominating set and $\gamma(G_n) = 1$.

Now in $V - D$, there exists a vertex v which is adjacent to all vertices since $< V - D >$ is complete. Define another set $D' = \{v\}$ of $V - D$ in G_n.

Then D' is also a dominating set of G_n , so D' is an inverse dominating set of G_n with respect to D and therefore $\gamma^{-1}(G_n) = 1$.
Hence $\gamma(G_n) = 1 = \gamma^{-1}(G_n)$. ■

Theorem 3.2: For a graph G_n, if $n = p_1 . p_2$ where p_1 and p_2 are primes, the inverse domination number

CAHIERS MAGELLANES-NS
Volume 06 Issue 1
2024

ISSN:1624-1940
DOI 10.6084/m9.figshare.26312694
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$\gamma^{-1}(G_n) = 2$.

Proof: Consider the graph G_n, with $n = p_1.p_2$ where p_1 and p_2 are primes.

Case 1: Let $p_1 = 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into two subsets as $S_1 = \{u: u = kp_1 < n, where\ k \in N\}$ and $S_2 = \{v: v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = p_2 - 1$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.

Then $D = \{u, v\}$ forms a dominating set and and $\gamma(G_n) = 2$.

Now in $V - D$, there exists a vertex u_1 which is an even numbered vertex of S_1 adjacent to every vertex of S_1 and every even numbered vertex of S_2 . Also there exists a vertex v_1 of S_2 is adjacent to remaining all vertices in S_2.

Define another set $D' = \{u_1, v_1\}$ of $V - D$ in G_n.

Then D' is also a dominating set of G_n , so D' is an inverse dominating set of G_n with respect to D and therefore $\gamma^{-1}(G_n) = 2$.

Hence $\gamma(G_n) = 2 = \gamma^{-1}(G_n)$. ■

Case 2: Let $p_1 > 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into two subsets as $S_1 = \{u: u = kp_1 < n, where\ k \in N\}$ and $S_2 = \{v: v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = \frac{2p2+p1-5}{2} + \lfloor\frac{p1-1}{3}\rfloor - \lfloor\frac{p2-1}{6}\rfloor$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.

Then $D = \{u, v\}$ forms a dominating set and and and $\gamma(G_n) = 2$.

Now in $V - D$, there exists a vertex u_1 which is an even numbered vertex of S_1 adjacent to every vertex

CAHIERS MAGELLANES-NS
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2024
ISSN:1624-1940
DOI 10.6084/m9.figshare.26312694
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of S_1 and every even numbered vertex of S_2 . Also there exists a vertex v_1 of S_2 is adjacent to remaining all vertices in S_2.

Define another set $D' = \{u_1, v_1\}$ of $V - D$ in G_n.

Then D' is also a dominating set of G_n , so D' is an inverse dominating set of G_n with respect to D and therefore $\gamma^{-1}(G_n) = 2$.

Hence $\gamma(G_n) = 2 = \gamma^{-1}(G_n)$.

Theorem 3.3: For a graph G_n, if $n = p_1{}^{a1} . p_2{}^{a2} \dots \dots . p_m{}^{am}$ where $p_1, p_2, \dots . p_m$ are primes and $a_1, a_2, \dots , a_m$ are natural numbers, the inverse domination number $\gamma^{-1}(G_n) = m$.

Proof: Consider the graph G_n, with $n = p_1{}^{a1} . p_2{}^{a2} \dots \dots . p_m{}^{am}$ where $p_1, p_2, \dots . p_m$ are primes and $a_1, a_2, \dots , a_m \in N$.

Then for the graph G_n, the vertex set V contains all the multiples of $p_1, p_2, \dots , p_m$ which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into $m-$ subsets as $S_1, S_2, \dots \dots , S_m$ where $S_i = \{v_i : v_i = k_i p_i < n, where\ k_i \in N\}$ for $i = 1\ to\ m$.

Then vertex $p_1 . p_2 \dots . p_m$ is a common vertex of all the $S_i's$ and it is adjacent to every vertex in the graph. Therefore a set D= $\{p_1 . p_2 \dots . p_m, \}$ forms a dominating set and and $\gamma(G_n) = 1$.

Now in $V - D$, there exists a vertex p_i which is adjacent to all its multiples, and the define $D' = \{p_1, p_2, \dots \dots p_m\}$ of $V - D$ in G_n.

Then D' is also a dominating set of G_n , so D' is an inverse dominating set of G_n with respect to D and therefore $\gamma^{-1}(G_n) = m$.

Hence $\gamma(G_n) = 1\ and\ \gamma^{-1}(G_n) = m$.

4. INVERSE TOTAL DOMINATION IN A GRAPH G_n

The concept of inverse total domination in graphs was given in Kulli et al. [11] and the necessaryand sufficient condition for the inverse total domination was studied along with the lower and upper bounds. The inverse total domination number of some special classes of graphs was studied in [38]. In this section, results on inverse total domination of an undirected graph G_n is presented and inverse total domination number of $G_{m,n}$ is obtained for various values of m, n.

Definition: Let $G(V, E)$ be a graph and D be a minimum total dominating set of G. If $V - D$ contains a total dominating set D' of G then D' is called an inverse dominating set of G with respect to D. The order of the smallest inverse total dominating set of G is called its inverse total domination number and

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ISSN:1624-1940
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it is denoted by $\gamma^{-1}(G)$.

Theorem 4.1: For a graph G_n, the inverse total domination number $\gamma_t^{-1}(G_n) = 2$, if $n = p^m$, where p is prime.

Proof: Consider the graph G_n, where $n = p^m$, and p is prime. Then the vertex set of graph G_n consisting of all multiples of p which are less than n. Set $V = \{p, 2p, 3p, ..., (p^{m-1} - 1)p\}$ and $|V| = p^{m-1} - 1$.

By the definition of graph G_n, it forms a complete graph and degree of each vertex is $|V| - 1$,which is maximum. Therefore every vertex $v_i = p$ in V is adjacent to all $v_j = 2p$ where $i \neq j$ of G_n as $\gcd(p, 2p) > 1$. Then $T = \{v_i, v_j\}$ forms a total dominating set and $\gamma_t(G_n) = 2$.

Since the graph G_n is complete and each vertex degree is $p^{m-1} - 2$, there exists a vertex whichis adjacent to all other vertices in the graph G_n.

Define another set $T' = \{3p, 4p\}$ from $V - D$ of G_n.

It implies all the vertices in $V - T$ are adjacent to some vertex in $T'$and therefore the induced subgraph $\langle T' \rangle$ has no isolated vertices.

Hence T'is an inverse total dominating set of $G_n$with respect to T and therefore $\gamma_t^{-1}(G_n) = 2$.

Theorem 4.2: For a graph G_n , if $n = p_1.p_2$ where p_1 and p_2 are primes, the inverse totaldomination number $\gamma_t^{-1}(G_n) = 2$.

Proof: Consider the graph G_n, with $n = p_1.p_2$ where p_1 and p_2 are primes.

Case 1: Let $p_1 = 3$. Then for the graph G_n, the vertex set V contains all p_1 –multiples and p_2 – multiples which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into two subsets as $S_1 = \{u: u = kp_1 < n, where\ k \in N\}$ and $S_2 = \{v: v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = p_2 - 1$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.

Then $T = \{u, v\}$ is a total dominating set and $\gamma_t(G_n) = 2$.

Choose vertex u_1 from $S_1 - \{u\}$ and v_1, an even numbered vertex in $S_2 - \{v\}$ of G_n.
Define another set $T' = \{u_1, v_1\}$ from $V - D$ of G_n. It implies all the vertices in $V - T$ areadjacent

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to some vertex in T' and therefore the induced subgraph $\langle T' \rangle$ has no isolated vertices.

Hence T' is an inverse total dominating set of G_n with respect to T and therefore $\gamma_t^{-1}(G_n) = 2$.

Case 2: Let $p_1 > 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into two subsets as $S_1 = \{u: u = kp_1 < n, where\ k \in N\}$ and $S_2 = \{v: v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = \frac{2p2+p1-5}{2} + \lfloor\frac{p1-1}{3}\rfloor - \lfloor\frac{p2-1}{6}\rfloor$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$. Then $T = \{u, v\}$ formsa total dominating set and $\gamma_t(G_n) = 2$. Choose vertex u_1 from $S_1 - \{u\}$ and v_1 , an even numbered vertex in $S_2 - \{v\}$ of G_n.

Define another set $T' = \{u_1, v_1\}$ from $V - D$ of G_n. It implies all the vertices in $V - T$ are adjacent to some vertex in T' and therefore the induced subgraph $\langle T' \rangle$ has no isolated vertices.

Hence T' is an inverse total dominating set of G_n with respect to T and therefore $\gamma_t^{-1}(G_n) = 2$.

Theorem 4.3: For a graph G_n, if $n = p_1^{a1} . p_2^{a2} \dots\dots . p_m^{am}$ where $p_1, p_2, \dots . p_m$ are primes and $a_1, a_2, \dots , a_m$ are natural numbers, the inverse total domination number $\gamma_t^{-1}(G_n) = m - 1$.

Proof: Consider the graph G_n, with $n = p_1^{a1} . p_2^{a2} \dots\dots . p_m^{am}$ where $p_1, p_2, \dots . p_m$ are primesand $a_1, a_2, \dots , a_m \in N$.

Then for the graph G_n, the vertex set V contains all the multiples of $p_1, p_2, \dots , p_m$ which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into $m -$ subsets as $S_1, S_2, \dots\dots\dots , S_m$ where $S_i = \{v_i: v_i = k_i p_i < n, where\ k_i \in N\}$ for $i = 1\ to\ m$. Then vertex $p_1 . p_2 \dots . p_m$ is a common vertex of all the $S_i's$ and it is adjacent to every vertex in the graph. Therefore a set $T = \{p_1 . p_2 \dots . p_m, p_m\}$ forms a total dominating set $\gamma_t(G_n) = 2$.

Now in $V - T$, there exists a vertex p_i which is adjacent to all its multiples. Define $T' = \{p_1, p_2, \dots\dots , p_{m-1}\}$ of $V - T$ in G_n.

Then T' is also a total dominating set of G_n , so T' is an inverse total dominating set of G_n with respect

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ISSN:1624-1940
DOI 10.6084/m9.figshare.26312694
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to T and therefore $\gamma^{-1}(G_n) = m - 1$.

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