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AN ANALYTICAL APPROACH OF THE WRONSKIAN METHOD FOR THE DISCUSSION OF THE SOLUTION OF CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract:
In this work, we introduce a method Wronskian for solving fractional differential equations with fractional derivatives defined by the conformable fractional derivatives. For illustrating the linear dependent or independent solution of fractional differential equations we use the Wronskian method. One of the greatest advantages of the wronskian is that it can be used with higher order differential equations, and so, for any nth order differential equation, as long as you know $(n-1)$ solutions, the wronskian aids in solving for the last general solution while adding information on the rest of them, such as linear To test the efficiency of this approach some illustrative examples are presented for verifying whether the solution of a FDE is linearly dependent or independent. Moreover, some properties of fractional Wronskian are discussed. In the non-homogeneous case, we present the Wronskian method to verify the solution of FDE.

Key Words: Fractional Wronskian, fractional differential equations, and fractional derivative.

Introduction:

Fractional differential equations (FDEs) have gained significant attention in recent years due to their ability to model complex real-life phenomena more accurately than traditional integer-order differential equations. These equations, which extend the concept of derivatives and integrals to non-integer orders, offer a powerful tool for describing processes that exhibit anomalous diffusion, memory effects, and spatial heterogeneity. As a result, FDEs have found applications across a wide range of disciplines, including Control Theory for Dynamic Systems, Electrical Networks, Engineering, Electrochemistry of Corrosion, Mechanics, Physics, and Biology [1, 2].

One of the notable advancements in this field is the development of conformable fractional calculus, which provides a more intuitive and flexible framework for fractional differentiation and integration. Conformable fractional derivatives retain many of the useful properties of integer-order derivatives, such as the product rule and chain rule, making them particularly appealing for both theoretical analysis and practical applications [3].

The Wronskian method, traditionally used to study the linear independence of solutions to differential equations, has proven to be a valuable tool in the context of conformable fractional differential equations (CFDEs). This method facilitates the examination of solution behaviors and the determination of conditions for the existence and uniqueness of solutions [4].

In this paper, we explore an analytical approach to the Wronskian method for the discussion of solutions to CFDEs. We begin by reviewing the fundamental concepts of fractional calculus and the conformable fractional derivative [5, 6]. Following this, we introduce the Wronskian determinant and demonstrate its application in the context of CFDEs. Through illustrative examples, we highlight the efficacy of the

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Wronskian method in analyzing the solution space of CFDEs and provide insights into the conditions under which unique solutions can be guaranteed [7, 8].

Our goal is to provide a comprehensive understanding of how the Wronskian method can be effectively employed to address the challenges posed by CFDEs. By leveraging this approach, researchers and practitioners can gain deeper insights into the behavior of fractional differential systems, paving the way for more accurate modeling and analysis of complex dynamic processes in various scientific and engineering domains [9, 10].

FDE: For $0 < \alpha \leq 1$, let D^{α} represents the α-CFD of y. Consider a type of FDE,

$$D^{\alpha}D^{\alpha}y + P(x)D^{\alpha} + Q(x) = 0 \qquad (1)$$

Fractional Wronskian: The Wronskian is a function that is used in the study of differential equations to determine whether a set of solutions is linearly independent. If the Wronskian is non-zero, the solutions are linearly independent; if it is zero, the solutions are linearly dependent. We have two functions $y_1(x)$as well as $y_2(x)$ is given below-

$$W_{\alpha}[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ D^{\alpha}y_1 & D^{\alpha}y_2 \end{vmatrix} \qquad (2),$$

For two functions $y_1(x)$ and $y_2(x)$ satisfying equation (2) and $0 < \alpha \leq 1$. The term α is the Wronskian of the function.

Let $y_1(x)\, y_2(x) y_3(x) \ldots\ldots\ldots\ldots\ldots\, y_n(x)\, y_{n+1}(x)$ the Wronskian is defined as the determinant of a matrix whose entries are these functions and their derivatives. For any $0 < \alpha \leq 1$, the determinant,

$$W_{\alpha}[\, y_1,\, y_2, \ldots\ldots\ldots\ldots\ldots\, y_n,\, y_{n+1}] = \begin{bmatrix} y_1 & y_2 & \cdots & \cdots & \cdots & \cdots & y_{n+1} \\ T_{\alpha}y_1 & T_{\alpha}y_2 & \cdots & \cdots & \cdots & \cdots & T_{\alpha}y_{n+1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ {}^{n-1}T_{\alpha}y_1 & {}^{n-1}T_{\alpha}y_2 & \cdots & \cdots & \cdots & \cdots & {}^{n-1}T_{\alpha}y_{n+1} \\ {}^{n}T_{\alpha}y_1 & {}^{n}T_{\alpha}y_2 & \cdots & \cdots & \cdots & \cdots & {}^{n}T_{\alpha}y_{n+1} \end{bmatrix} \qquad (3).$$

The equation you provided appears to be a generalized form of the Wronskian determinant, incorporating a parameter α and a transformation operator T_{α}. This form is sometimes referred to as a generalized Wronskian or generalized Wronskian determinant. The generalized Wronskian W_{α} is an extension of the classical Wronskian determinant, incorporating a parameterized transformation operator to examine the linear independence of functions under more general transformations.

Preliminaries:

Calculus was developed alongside the concept of the fractional derivative. Commonly used definitions "include:

a) Riemann-Liouville Definition: The Riemann-Liouville equation is associated with fractional calculus, specifically with fractional differential equations. The Riemann-Liouville operator is one of the most common definitions of a fractional derivative.

The Riemann-Liouville fractional integral of order α of a function $f(x)$ is defined as

$$D_a^{\alpha}\big(f(t)\big) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\,dx.$$

here, in the above equation n indicates a positive integer and $\alpha \in [n-1, n)$. The satisfied the condition as $t \geq a$.

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b) Caputo Definition: A fractional derivative is widely used in the context of fractional calculus. It is defined in a way that generalizes the concept of an integer-order derivative to non-integer (fractional) orders, the α derivative of f is for $\alpha \in [n-1, n]$,
$D_a^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$".
Nevertheless, the following are the disadvantages of various definitions.

i. if α is not a natural number, the Riemann- Liouville derivative does not fulfil $D_a^{\alpha}(1) = 0$.

ii. The known product rule is not fulfilled by every fractional derivative,
$$D_a^{\alpha}(fg) = fD_a^{\alpha}(g) + gD_a^{\alpha}(f)$$

iii. The known quotient rule is not fulfilled by every fractional derivative.
$$D_a^{\alpha}\left(\frac{f}{g}\right) = \frac{fD_a^{\alpha}(g) + gD_a^{\alpha}(f)}{g^2}$$

iv. All fractional derivatives do not satisfy the chain rule.
$$D_a^{\alpha}(fg) = f^{\alpha}(g(a)).g^{\alpha}(f(a))$$

a. Conformable fractional derivative: A new concept which is a conformable fractional derivative which defines as For $\alpha \in (0, 1]$, & $f: [0, \infty) \to R$, Suppose

$$T_{\alpha}f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon(t)^{1-\varepsilon}) - f(t)}{\varepsilon}$$

For all $t > 0, \alpha \in (0, 1)$. T_{α} is referred to conformable fractional derivative f of order α
This new definition fulfills the following properties –

i. $T_{\alpha}(af_1 + bf_2) = aT_{\alpha}(f_1) + bT_{\alpha}(f_2)$, for every $a, b \in R$.

ii. $T_{\alpha}(\lambda) = 0$, for every constant function $f(t) = \lambda$.

Further for $\alpha \in (0, 1]$ and f, g are α- differential at a point t along with $g(t) \neq 0$ then"

iii. $T_{\alpha}(f_1, f_2) = f_1T_{\alpha}(f_2) + f_2T_{\alpha}(f_1)$.

iv. $T_{\alpha}\left(\frac{f_1}{f_2}\right) = \frac{f_1T_{\alpha}(f_2) - f_2T_{\alpha}(f_1)}{f_2^2}$.

Conformable FDE: Conformable fractional calculus provides a relatively new approach to defining fractional derivatives, which preserves some of the key properties of integer-order calculus while generalizing to non-integer orders. In this section, we introduce a linear homogeneous conformable differential equation with variable coefficients. Let's consider a more general form of a conformable fractional differential equation, where the differential equation has variable coefficients and involves multiple orders of the conformable derivative. This type of equation can be written as:

$$\sum_{n=0}^{k} P_n(t)^{n}T_{\alpha}\, y(t) = 0 \qquad (4)$$

Here, $P_n(t)$ are variable coefficient functions and ${}^{n}T_{\alpha}$ denotes the k-th order conformable fractional derivative of $y(t)$. Expanding the equation (4), we get,

$$^{n+1}T_{\alpha}y + P_n(t)^{n}T_{\alpha}y + P_{n-1}(t)^{n-1}T_{\alpha}y + \cdots + P_2(t)^{2}T_{\alpha}y + P_1(t)T_{\alpha}y + P_0(t)y = 0 \qquad (5)$$

$$L_{\alpha}[y] = {}^{n+1}T_{\alpha}y + P_n(t)^{n}T_{\alpha}y + P_{n-1}(t)^{n-1}T_{\alpha}y + \cdots + P_2(t)^{2}T_{\alpha}y + P_1(t)T_{\alpha}y + P_0(t)y \quad (6)$$

Important Result:
To solve the non-homogeneous conformable fractional differential equation using the method of variation of parameters, let's delve into the details of the problem. Let y_1 and y_2 be two independent solutions for-
Given the homogeneous equation:

$$D^{\alpha}(D^{\alpha}y) + a_1D^{\alpha}y + a_0y = 0 \qquad (7)$$

and the non-homogeneous equation:

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$D^{\alpha}(D^{\alpha}y) + a_1 D^{\alpha}y + a_0 y = f(x)$ (8)

where D_{α} denotes the conformable fractional derivative of order α, and a_1, a_0 are constants. The particular solution $y_p(x)$ for the non-homogeneous equation (7) can be found using the solutions of the homogeneous equation (6), Using variation of parameters, the particular solution $y_p(x)$ is given by:

$$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W^{\alpha}[y_1,y_2]x^{1-\alpha}} dx + y_2 \int \frac{y_1 f(x)}{W^{\alpha}[y_1,y_2]x^{1-\alpha}} dx \quad (9)$$

Theorem:

Let $y_1(t), y_2(t), \ldots\ldots\ldots\ldots . y_n(t), y_{n+1}(t)$ be $(n+1)$ solutions of $L_{\alpha}[y]$. If there is a $t_0 \in (a,b)$ then $\{y_1, y_2, \ldots\ldots\ldots\ldots\ldots\ldots . y_n, y_{n+1}\}$ indicates a fundamental set of solution if and only if $\{y_1, y_2, \ldots\ldots\ldots\ldots . y_n, y_{n+1}\} \neq 0$,

Verification: If $y(t)$ presents a solution $L_{\alpha}[y] = 0$ then we can write $y(t)$it as a linear combination of $\{y_1, y_2, \ldots\ldots\ldots\ldots\ldots\ldots . y_n, y_{n+1}\}$ that is –

$y = c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots\ldots + c_{n+1} y_{n+1}$ (10)

Now to find the constants, $c_k, 1 \leq k \leq n+1$. This linear system of equations may be expressed as follows:

$c_1 y_1(t_0) + c_2 y_2(t_0) + \ldots\ldots\ldots\ldots + c_{n+1} y_{n+1}(t_0) = y(t_0)$

$c_1 T_{\alpha} y_1(t_0) + c_2 T_{\alpha} y_2(t_0) + \ldots\ldots\ldots\ldots + c_{n+1} T_{\alpha} y_{n+1}(t_0) = T_{\alpha} y(t_0)$

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$c_1 {}^{n}T_{\alpha} y_1(t_0) + c_2 {}^{n}T_{\alpha} y_2(t_0) + \ldots\ldots\ldots\ldots + c_{n+1} {}^{n}T_{\alpha} y_{n+1}(t_0) = {}^{n}T_{\alpha} y(t_0)$

Applying Cramer's rule, we have

$c_k = \frac{W_{\alpha}^{k}}{W_{\alpha}} \quad 1 \leq k \leq n+1$ (11)

Since the constant $c_1, c_2, \ldots\ldots\ldots\ldots . c_n, c_{n+1}$ exist so we must have $W_{\alpha}(t) \neq 0$. Hence, $W(y_1, y_2, \ldots\ldots\ldots\ldots . y_n, y_{n+1}) \neq 0$

Application of Wronskian:

We will discuss the nature of the solution of the FDE using Wronskian also we will discuss the properties and theorems of conformable FDEs using" Wronskian.

Examples:

Let $y_1 \, and \, y_2$ are the solution of a FDE $D^{\frac{1}{2}} D^{\frac{1}{2}} y + \sqrt{x} D^{\frac{1}{2}} y = 0$ then $y_1 \, and \, y_2$ are the independent solution.

Solution: Comparing the fractional differential equation,

$D^{\frac{1}{2}} D^{\frac{1}{2}} y + \sqrt{x} D^{\frac{1}{2}} y = 0$ with (12)

$$D^{\alpha} D^{\alpha} y + P(x) D^{\alpha} + Q(x) = 0$$

we get, $P(x) = \sqrt{x}$ and Q(x) = 0

Now, Wronskian of the solutions $y_1 \, and \, y_2$ is given by,

$$W_{\frac{1}{2}}[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ D^{\frac{1}{2}} y_1 & D^{\frac{1}{2}} y_2 \end{vmatrix}$$

$$= y_1 D^{\frac{1}{2}} y_2 - y_2 D^{\frac{1}{2}} y_1$$

Now, $D^{\frac{1}{2}} \left(W_{\frac{1}{2}}[y_1, y_2] \right) = D^{\frac{1}{2}} \left(y_1 D^{\frac{1}{2}} y_2 - y_2 D^{\frac{1}{2}} y_1 \right)$ (13)

Since y_1 and y_2 are the solution of (3) then we get,

$D^{\frac{1}{2}} D^{\frac{1}{2}} y_1 = -\sqrt{x} D^{\frac{1}{2}} y_1$ (14a)

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$D^{\frac{1}{2}}D^{\frac{1}{2}}y_2 = -\sqrt{x}D^{\frac{1}{2}}y_2$ (14b)

Substituting in (4) we get,

$$D^{\frac{1}{2}}\left(W_{\frac{1}{2}}[y_1, y_2]\right) = D^{\frac{1}{2}}\left(y_1 D^{\frac{1}{2}}y_2 - y_2 D^{\frac{1}{2}}y_1\right)$$

$$= D^{\frac{1}{2}}\, y_1\, D^{\frac{1}{2}}\, y_2 + y_1 D^{\frac{1}{2}}\, D^{\frac{1}{2}}\, y_2 - D^{\frac{1}{2}}\, y_2\, D^{\frac{1}{2}}\, y_1 - \, y_2 D^{\frac{1}{2}}\, D^{\frac{1}{2}}\, y_1$$

$$= -\sqrt{x}\left(y_1 D^{\frac{1}{2}}y_2 - y_2 D^{\frac{1}{2}}y_1\right)$$

$$= -\sqrt{x}\left(W_{\frac{1}{2}}[y_1, y_2]\right)$$

$$\Rightarrow \frac{D^{\frac{1}{2}}\left(W_{\frac{1}{2}}[y_1, y_2]\right)}{W_{\frac{1}{2}}[y_1, y_2]} = -\sqrt{x} \quad (15)$$

Integrating both sides and we get,

$W_{\frac{1}{2}}[y_1, y_2] = I_{\frac{1}{2}}(\sqrt{x}) \neq 0, \alpha > 0$ (16)

Therefore y_1 *and* y_2 are independent solutions.

So $\{y_1,\ y_2,\}$ forms a fundamental set of solutions.

Let s_1 and s_2 are the solution of a FDE given below then s_1 and s_2 are the linearly independent solution-

$D^{\frac{1}{2}}s = \left(-s + t^2 + \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}}\right)$ (17)

Solution: Given the equation,

$$\Rightarrow D^{\frac{1}{2}}s = \left(-s + t^2 + \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}}\right)$$

$$\Rightarrow D^{\frac{1}{2}}D^{\frac{1}{2}}s = D^{\frac{1}{2}}\left(-s + t^2 + \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}}\right)$$

$\Rightarrow Ds = D^{\frac{1}{2}}(-s) + D^{\frac{1}{2}}(t^2) + \frac{8}{3\sqrt{\pi}}\, D^{\frac{1}{2}}t^{\frac{3}{2}}$ (18)

$g` = A + B + C$ (19)

Now, from the eqution (17), and comparing the equation (18) and (19), we got

$$A = D^{\frac{1}{2}}(-s) = -D^{\frac{1}{2}}(s) = -\left(-s + t^2 + \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}}\right)$$

$= s - t^2 - \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}}$ (20a)

$$B = D^{\frac{1}{2}}(t^2)$$

$= \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}}$ (20b)

$$C = \frac{8}{3\sqrt{\pi}}\, D^{\frac{1}{2}}t^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}} \cdot \frac{3\sqrt{\pi}}{4} t$$

$= 2t$ (20c)

So from equation (19) and putting the equations (20a), (20b), and (20c), then we get,

$$g` = A + B + C$$

$$= s - t^2 - \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}} + \frac{8}{3\sqrt{\pi}}\, t^{\frac{3}{2}} + 2t$$

$= s + 2t - t^2$ (21) We

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can write, $(s-t^2)+2t = s-t^2+2t$ (22)
So the general solution of the differential equation (22), $s = t^2 + Ce^t$ (23)
Now, we are going to check whether the solutions are linearly independent or dependent,
Let, $\phi_1 = t^2$ and $\phi_1 = e^t$

$$W_{\frac{1}{2}}[t^2, e^t] = \begin{vmatrix} t^2 & e^t \\ D^{\frac{1}{2}}t^2 & D^{\frac{1}{2}}e^t \end{vmatrix} = \begin{vmatrix} t^2 & e^t \\ 2t^{\frac{3}{2}} & \sqrt{t}e^t \end{vmatrix}$$

$= t^{\frac{3}{2}}e^t(t-2) \neq 0$ (24)
Hence, from the equation (24), can said that the solutions ϕ_1 and ϕ_2 are linearly independent and $\{s_1, s_2\}$ are a fundamental set of solutions.

The solutions of the FDE: In this section, we explore the solutions to a conformable fractional differential equation (FDE) using the fractional Wronskian method.

$D^{\frac{1}{2}}D^{\frac{1}{2}}y - \sqrt{x}\,y = 0$, are linearly dependent.

Solution: Let y_1 *and* y_2 are the solution of the given FDE,
$D^{\frac{1}{2}}D^{\frac{1}{2}}y - \sqrt{x}\,y = 0$ (25)
Now, $W_\alpha[y_1, y_2] = y_1 D^\alpha y_2 - y_2 D^\alpha y_1$

$$D^\alpha W_\alpha[y_1, y_2] = D^\alpha(y_1 D^\alpha y_2 - y_2 D^\alpha y_1)$$

$$= D^\alpha\, y_1\, D^\alpha\, y_2 - y_1 D^\alpha\, D^\alpha\, y_2 - D^\alpha\, y_2\, D^\alpha\, y_1 + y_2 D^\alpha D^\alpha\, y_1$$

$= y_2 D^\alpha D^\alpha\, y_1 - y_1 D^\alpha\, D^\alpha\, y_2$ (26)
Since y_1 and y_1 be the solution of (25) so
$D^{\frac{1}{2}}D^{\frac{1}{2}}y = \sqrt{x}\,y_1$ and $D^{\frac{1}{2}}D^{\frac{1}{2}}y = \sqrt{x}\,y_2$
Now from (26) we have,

$$D^\alpha W_\alpha[y_1, y_2] = y_2\sqrt{x}\,y_1 - y_1\sqrt{x}\,y_2 = 0$$

Or, $\frac{D^\alpha W_\alpha[y_1,y_2]}{W_\alpha[y_1,y_2]} = 0$
Using fractional integrating we get,
$W_\alpha[y_1, y_2] = I_\alpha\,(0\,) = 0, \quad \alpha > 0$
Therefore y_1 *and* y_2 are dependent solutions.
Example of the solutions of the fractional differential equation is given below
$y^{\frac{1}{2}} + y = x^2 + 2x^{\frac{3}{2}}$, are dependent.
Let the solution of the homogeneous equation $y^{\frac{1}{2}} + y = 0$ is of the for
$y_h = e^{r\sqrt{x}}$.
Now, $y^{\frac{1}{2}} + y = 0$

$$\Rightarrow \frac{r}{2}e^{r\sqrt{x}} + e^{r\sqrt{x}} = 0$$

The auxiliary equation is, $\frac{1}{2}\,r + 1 = 0 \Rightarrow r = -2$
Hence, $y_h = e^{-2\sqrt{x}}$
It is easy to examine that, $y_p = x^2$ is a particular solution of the nonhomogeneous equation.
So the general solution of the equation is,

$y(x) = y_h(x) + y_p(x) \Rightarrow y(x) = Ae^{-2\sqrt{x}} + Bx^2$,
where A and B are constant can be determined using boundary conditions.

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In order to check the solution of this fractional differential equation are dependent or independent solution let us consider,
$\phi_1(x) = e^{-2\sqrt{x}}$ and $\phi_2(x) = x^2$
Now, $W[\phi_1, \phi_2] = W[e^{-2\sqrt{x}}, x^2] \neq 0$
So, we can said that $W[\phi_1, \phi_2] \neq 0$,
Hence the solution function $\phi_1, and\ \phi_2$are linearly independent solution
And so $\{y_1, y_2,\}$ forms a fundamental set of solutions.

Conclusion:
In our work, we have extended the classical Wronskian method to the domain of fractional differential equations (FDEs). By introducing the concept of a fractional Wronskian's, we have been able to demonstrate the fundamental set of solutions for an FDE. Our discussion includes various properties of the fractional Wronskian, highlighting its utility in analyzing the solutions of fractional differential equations.
The fractional Wronskian's approach provides a reliable method for determining the dependence or independence of solutions to an FDE. Through our analysis, we have shown that the behavior and results derived using the fractional Wronskian are consistent with those obtained from classical differential equations. This analogy not only reinforces the validity of the fractional Wronskian method but also emphasizes its significance in the broader context of fractional calculus.
Our findings indicate that the fractional Wronskian retains the essential characteristics of its classical counterpart, making it a powerful tool in the study of fractional differential equations. This work opens up new avenues for the application of fractional calculus in various scientific and engineering fields, where understanding the fundamental nature of solutions is crucial.

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